

Triangular Numbers



Teacher Notes & Answers

7 8 9 10 11 **12**



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Activity



Student



30 min

Introduction to Induction

The focus of this investigation to determine a rule for the sum of the first n whole numbers, initially by observation (slide presentation) and finally a proof that the formula is true for all values of n .

Visual Observation

Question: 1.

The following questions refer to the last pattern ($n = 6$).

- How many dots in the last pattern?
- Explain how you determined this quantity.
- What is the sum of the first 6 whole numbers?

Question: 2.

Determine the sum of the first 7 whole numbers without using 'addition'.

Question: 3.

Generalise your answer to Question 2 for the sum of the first n whole numbers.

Question: 4.

Use your formula (above) to calculate the sum of the first 100 whole numbers.

Question: 5.

Use the sum command on your calculator to determine the sum of the first 100 numbers.

Expression: $\sum_{x=1}^{100} x$ Calculator instructions: Press **[math]** and select option: **5** sum(.

Visual Observation + Numerical Intuition:

In this section you will study the sum of the first n odd numbers, then the first n even numbers and finally derive a formula for the sum of the first n whole numbers.

Question: 6.

The following questions relate to the sum of the first n odd numbers.

- What shape can be formed by the sum of the first n odd numbers?
- Write a formula for the sum of the first n odd numbers.
- Use your calculator to check the sum of the first 50 odd numbers by using: $\sum_{x=1}^{50} 2x - 1$

Question: 7.

The sum of the first n even numbers can be determined by comparing with the sum of the first n odd numbers.

$$1 + 3 + 5 + 7 + 9 + 11 + 13 \dots$$

$$2 + 4 + 6 + 8 + 10 + 12 + 14 \dots$$

Notice that each of the even numbers is '1' more than the corresponding odd number.

- Based on this information, determine the sum of the first 50 even numbers.
- Write a formula for the sum of the first n even numbers.
- Use your calculator to check the sum of the first 50 even numbers by using: $\sum_{x=1}^{50} 2x$
- If sum of the first n even numbers: $2 + 4 + 6 + \dots$ is divided 2, the result is $1 + 2 + 3 \dots$ hence write a formula for the sum of the first n whole numbers.

Pascal's Triangle – Hidden Gem

Pascal's triangle also contains the triangular numbers.

Notice: The n^{th} triangular number is in the $(n+1)^{\text{th}}$ row¹.

Example: The number 15 is the 5th triangular number and it is located in the 6th row.

Recall that the elements in Pascal's triangle can be computed

using combinatorics: ${}^nC_r = \frac{n!}{(n-r)!r!}$

					1					
					1		1			
					1		2		1	
					1		3		3	
					1		4		6	
					1		5		10	
					1		6		15	
					1		7		21	
					1		8		28	
					1		9		36	
					1		10		45	
					1		11		55	
					1		12		66	
					1		13		78	
					1		14		91	
					1		15		105	
					1		16		120	
					1		17		136	
					1		18		153	
					1		19		171	
					1		20		190	
					1		21		210	
					1		22		231	
					1		23		253	
					1		24		276	
					1		25		300	
					1		26		325	
					1		27		351	
					1		28		378	
					1		29		406	
					1		30		435	
					1		31		465	
					1		32		496	
					1		33		528	
					1		34		561	
					1		35		595	
					1		36		630	
					1		37		666	
					1		38		703	
					1		39		741	
					1		40		780	
					1		41		820	
					1		42		861	
					1		43		903	
					1		44		946	
					1		45		990	
					1		46		1035	
					1		47		1081	
					1		48		1128	
					1		49		1176	
					1		50		1225	
					1		51		1275	
					1		52		1326	
					1		53		1378	
					1		54		1431	
					1		55		1485	
					1		56		1540	
					1		57		1596	
					1		58		1653	
					1		59		1711	
					1		60		1770	
					1		61		1830	
					1		62		1891	
					1		63		1953	
					1		64		2016	
					1		65		2080	
					1		66		2145	
					1		67		2211	
					1		68		2278	
					1		69		2346	
					1		70		2415	
					1		71		2485	
					1		72		2556	
					1		73		2628	
					1		74		2701	
					1		75		2775	
					1		76		2850	
					1		77		2926	
					1		78		3003	
					1		79		3081	
					1		80		3160	
					1		81		3240	
					1		82		3321	
					1		83		3403	
					1		84		3486	
					1		85		3570	
					1		86		3655	
					1		87		3741	
					1		88		3828	
					1		89		3916	
					1		90		4005	
					1		91		4095	
					1		92		4186	
					1		93		4278	
					1		94		4371	
					1		95		4465	
					1		96		4560	
					1		97		4656	
					1		98		4753	
					1		99		4851	
					1		100		4950	
					1		101		5050	
					1		102		5151	
					1		103		5253	
					1		104		5356	
					1		105		5460	
					1		106		5565	
					1		107		5671	
					1		108		5778	
					1		109		5886	
					1		110		5995	
					1		111		6105	
					1		112		6216	
					1		113		6328	
					1		114		6441	
					1		115		6555	
					1		116		6670	
					1		117		6786	
					1		118		6903	
					1		119		7021	
					1		120		7140	
					1		121		7260	
					1		122		7381	
					1		123		7503	
					1		124		7626	
					1		125		7750	
					1		126		7875	
					1		127		8001	
					1		128		8128	
					1		129		8256	
					1		130		8385	
					1		131		8515	
					1		132		8646	
					1		133		8778	
					1		134		8911	
					1		135		9045	
					1		136		9180	
					1		137		9316	
					1		138		9453	
					1		139		9591	
					1		140		9730	
					1		141		9870	
					1		142		10011	
					1		143		10153	
					1		144		10296	
					1		145		10440	
					1		146		10585	
					1		147		10731	
					1		148		10878	
					1		149		11026	
					1		150		11175	
					1		151		11325	
					1		152		11476	
					1		153		11628	
					1		154		11781	
					1		155		11935	
					1		156		12090	
					1		157		12246	
					1		158		12403	
					1		159		12561	
					1		160		12720	
					1		161		12880	
					1		162		13041	
					1		163		13203	
					1		164		13366	
					1		165		13530	
					1		166		13695	
					1		167		13861	
					1		168		14028	
					1		169		14196	
					1		170		14365	
					1		171		14535	
					1		172		14706	
					1		173		14878	
					1		174		15051	
					1		175		15225	
					1		176		15400	
					1		177		15576	
					1		178		15753	
					1		179		15931	
					1		180		16110	
					1		181		16290	
					1		182		16471	
					1		183		16653	
					1		184		16836	
					1		185		17020	
					1		186		17205	
					1		187		17391	
					1		188		17578	
					1		189		17766	
					1		190		17955	
					1		191		18145	
					1		192		18336	
					1		193		18528	
					1		194		18721	
					1		195		18915	
					1		196		19110	
					1		197		19306	
					1		198		19503	
					1		199		19701	
					1		200		19900	
					1		201		20100	
					1		202		20301	
					1		203		20503	
					1		204		20706	
					1		205		20910	
					1		206		21115	
					1		207		21321	
					1		208		21528	
					1		209		21735	
					1		210		21943	
					1		211			

Question: 8.

- Use your calculator to write down the first 9 triangular numbers. [Store the values in L_2]
- Use combinatorics to calculate the 100th triangular number, the sum of the first 100 whole numbers.
- Store the numbers: $\{1, 2, 3 \dots 9\}$ in L_1 and use Quadratic regression to determine an equation relating L_1 to L_2 . Write down the regression equation. [Calculator Instructions Below]



The equation stored in $f(x)$ can be tested by generating a table of values.

Press the **[Table]** key and select option 1: Add/Edit func. The function will be displayed.

Use the arrow keys to navigate down through the menu to 'calc' then press **[Enter]** to generate the table.

- The diagonal for the triangular numbers can be written using combinatorics: ${}^{n+1}C_2 = \frac{(n+1)!}{((n+1)-2)!2!}$

Simplify this formula to write an expression for the n^{th} triangular number.

¹ Row numbering in Pascal's triangle starts at row(0) = {1}, row(1) = {1, 1}, row(2) = {1, 2, 1}

Induction

The formula for the sum of the first n whole numbers has been generated three different ways. In each case the formula has been based on 'observation', not proof.

Step 1: Show true for $n = 1$.

We must first prove that the formula is true for $n = 1$.

The sum of the first '1' whole numbers is equal to 1 and $\frac{n(n+1)}{2} = \frac{1 \times 2}{2} = 1$

Step 2: Assume true for n

That is: $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ -- Equation 1

Step 3: Show true for $n + 1$.

We know that the LHS = $(1 + 2 + 3 + \dots + n) + (n + 1)$,

From Equation 1 we can re-write this as: $\frac{n(n+1)}{2} + (n + 1)$

Question: 9.

Complete step 3 by re-writing the RHS: $\frac{(n+1)(n+1+1)}{2}$ to show that: $\frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+1+1)}{2}$