Triangular Numbers

Teacher Notes & Answers

7 8 9 10 11 **12**

TI-30XPlus MathPrint™ | Activity | Student | 30 min

Introduction to Induction

The focus of this investigation to determine a rule for the sum of the first *n* whole numbers, initially by observation (slide presentation) and finally a proof that the formula is true for all values of n.

Visual Observation

Question: 1.

The following questions refer to the last pattern (n = 6).

a) How many dots in the last pattern?

b) Explain how you determined this quantity.

c) What is the sum of the first 6 whole numbers?

Question: 2.

Determine the sum of the first 7 whole numbers without using 'addition'.

Question: 3.

Generalise your answer to Question 2 for the sum of the first *n* whole numbers.

Question: 4.

Use your formula (above) to calculate the sum of the first 100 whole numbers.

Question: 5.

Use the sum command on your calculator to determine the sum of the first 100 numbers.

Expression: $\sum_{x=1}^{100} x$ Calculator instructions: Press **[math]** and select option: **5** sum(.

Visual Observation + Numerical Intuition:

In this section you will study the sum of the first *n* odd numbers, then the first *n* even numbers and finally derive a formula for the sum of the first *n* whole numbers.

Question: 6.

The following questions relate to the sum of the first *n* odd numbers.

a) What shape can be formed by the sum of the first *n* odd numbers?

b) Write a formula for the sum of the first *n* odd umbers.

c) Use your calculator to check the sum of the first 50 odd numbers by using: $\sum_{x=1}^{50} 2x-1$

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Question: 7.

The sum of the first n even numbers can be determined by comparing with the sum of the first n odd numbers.

1 + 3 + 5 + 7 + 9 + 11 + 13 …

2 + 4 + 6 + 8 + 10 + 12 + 14 …

Notice that each of the even numbers is '1' more than the corresponding odd number.

a) Based on this information, determine the sum of the first 50 even numbers.

b) Write a formula for the sum of the first *n* even numbers.

c) Use your calculator to check the sum of the first 50 even numbers by using: $\sum_{x=1}^{50} 2x$

d) If sum of the first *n* even numbers: 2 + 4 + 6 + …is divided 2, the result is 1 + 2 + 3 … hence write a formula for the sum of the first n whole numbers.

Pascal's Triangle – Hidden Gem

Pascal's triangle also contains the triangular numbers.

Notice: The n^{th} triangular number is in the $(n+1)^{th}$ row[1].

Example: The number 15 is the 5^{th} triangular number and it is located in the 6^{th} row.

Recall that the elements in Pascal's triangle can be computed using combinatorics: $^{n}C_{r} = \dfrac{n!}{(n-r)!r!}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Question: 8.

a) Use your calculator to write down the first 9 triangular numbers. [Store the values in L_2]

b) Use combinatorics to calculate the 100^{th} triangular number, the sum of the first 100 whole numbers.

c) Store the numbers: {1, 2, 3... 9} in L_1 and use Quadratic regression to determine an equation relating L_1 to L_2. Write down the regression equation. [Calculator Instructions Below]

The equation stored in $f(x)$ can be tested by generating a table of values.

Press the **[Table]** key and select option 1: Add/Edit func. The function will be displayed.

Use the arrow keys to navigate down through the menu to 'calc' then press **[Enter]** to generate the table.

d) The diagonal for the triangular numbers can be written using combinatorics: $^{n+1}C_{2} = \dfrac{(n+1)!}{((n+1)-2!)2!}$

Simplify this formula to write an expression for the n^{th} triangular number.

[1] Row numbering in Pascal's triangle starts at row(0) = {1}, row(1) = {1, 1}, row(2) = {1, 2, 1}

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Induction

The formula for the sum of the first n whole numbers has been generated three different ways. In each case the formula has been based on 'observation', not proof.

Step 1: Show true for n = 1.

We must first prove that the formula is true for n = 1.

The sum of the first '1' whole numbers is equal to 1 and $\frac{n(n+1)}{2} = \frac{1\times 2}{2} = 1$

Step 2: Assume true for n

That is: 1 + 2 + 3 + … n = $\frac{n(n+1)}{2}$ **-- Equation 1**

Step 3: Show true for n + 1.

We know that the LHS = (1 + 2 + 3 + … n) + (n + 1),

From Equation 1 we can re-write this as: $\frac{n(n+1)}{2} + (n+1)$

Question: 9.

Complete step 3 by re-writing the RHS: $\frac{(n+1)(n+1+1)}{2}$ to show that: $\frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+1+1)}{2}$

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